Indian Statistical Institute, Bangalore Centre
B.Math. (I Year) : 2010-2011
Semester II : Mid-Semestral Examination
Probability Theory II

2.3.2011 Time: $2\frac{1}{2}$ hours. Maximum Marks : 80

Note: The paper carries 85 marks. Any score above 80 will be taken as 80. State clearly the results you are using in your answers.

1. [10+10=20 marks] Prove or disprove the following:

(i) There is a constant C such that the function f given by

$$f(x,y) = C\exp\{-\frac{1}{2}(x^2 - 2xy + y^2)\},\ (x,y) \in \mathbb{R}^2,$$

is a probability density function on $\mathbb{R}^2$.

(ii) Let (X,Y) be a two dimensional random variable having a bivariate normal distribution. Then X,Y are independent if and only if $\mathrm{Cov}(X,Y)$ $= 0$.

2. [7+8+15=30 marks] Let (X,Y) be a two dimensional absolutely continuous random variable with probability density function

$$\begin{aligned} f(x,y) &= C(y-x), \text{ if } 0 < x < y < 1, \\ &= 0, \text{ otherwise} \end{aligned}$$

where C is a constant.

(i) Find the value of C.

(ii) Find the marginal probability density functions.

(iii) Find the probability density function of $Z = X + Y$. (It may be a good idea to check if your answer is indeed a probability density function.)

3. [10+10=20 marks] (i) Let (X,Y) be a two dimensional absolutely continuous random variable with probability density function $f(\cdot,\cdot)$. Find the probability density function of the random variable $\frac{Y}{X}$ in terms of $f(\cdot,\cdot)$.

(ii) Let X,Y be independent random variables having exponential distributions with parameters λ_1, λ_2 respectively. Find the probability density function of $\frac{Y}{X}$.

4. [10+5=15 marks] Let $X_1, \cdots, X_n$ be independent identically distributed (i.i.d.) $N(0, \sigma^2)$ random variables. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation such that $||A\underline{x}|| = ||\underline{x}||$ for all $\underline{x} \in \mathbb{R}^n$, and $|\det A| = 1$. Denote $\underline{X} = (X_1, \cdots, X_n)^t$. Let $\underline{Y} = (Y_1, \cdots, Y_n)^t$ be given by $\underline{Y} = A\underline{X}$. (Note that $||\underline{z}|| = \sqrt{\sum_{i=1}^n |z_i|^2}$ for $\underline{z} \in \mathbb{R}^n$.)

(i) Find the joint probability density function of $Y_1, \cdots, Y_n$.

(ii) What is the covariance matrix of $Y_1, \cdots, Y_n$?